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                BOUNDS FOR OPTIMAL CONFIDENCE LIMITS FOR SERIES SYSTEMS.(U)
                JUN 80   B HARRIS, A P SOMS
UNCLASSIFIED   MRC-TSR-2093   N00014-79-C-0321
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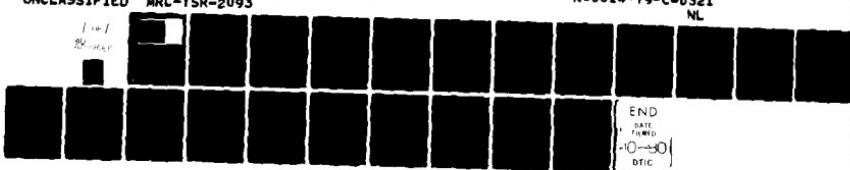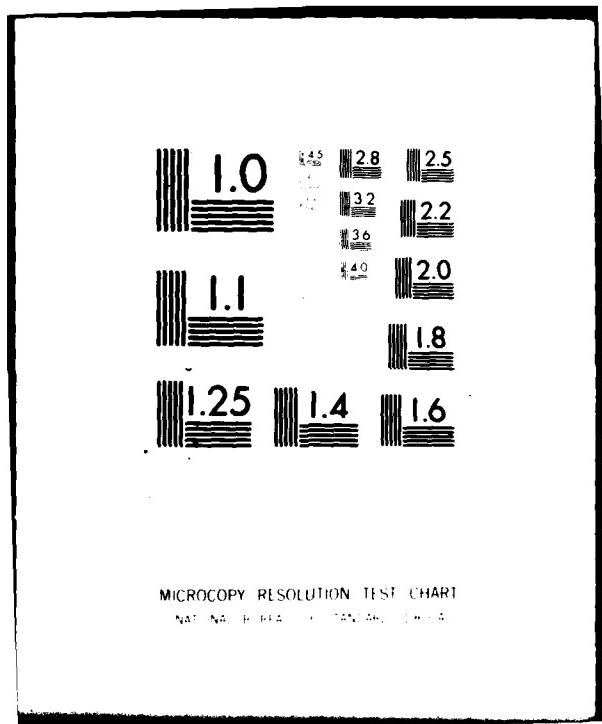
MRC Technical Summary Report # 2093

BOUNDS FOR OPTIMAL CONFIDENCE
LIMITS FOR SERIES SYSTEMS

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June 1980

(Received May 21, 1980)

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

Lindstrom-Madden type approximations to the lower
confidence limit on the reliability of a series system are
theoretically justified by extending and simplifying the
results of Sudakov (1973). Applications are made to Johns
(1976) and Winterbottom (1974). Numerical examples are
presented.

AMS (MOS) Subject Classifications: 62N05, 90B25.

Key Words: Lindstrom-Madden approximation; Optimal
confidence bounds; Reliability; Series
system.

Work Unit Number 4 - Statistics and Probability

SIGNIFICANCE AND EXPLANATION

Serial systems arise naturally in practice in engineering and physics. Therefore it is of substantial significance to be able to efficiently utilize data obtained on individual components for the purpose of obtaining an overall assessment of the reliability of the system. The methods of this paper may be employed for this purpose.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

BOUNDS FOR OPTIMAL CONFIDENCE
LIMITS FOR SERIES SYSTEMS

Bernard Harris and Andrew P. Soms[*]

1. Introduction and Summary

A problem of fundamental interest to practitioners in
reliability is the statistical estimation of the reliability of a
system using experimental data collected on subsystems. In this
paper, the subsystem data available consists of a sequence of
Bernoulli trials in which a "one" is recorded if the subsystem
functions and a zero is recorded if the subsystem fails. Thus
for each of the k subsystems composing the system, the data pro-
vided consists of the pair (n_i, Y_i), $i=1,2,\ldots k$, where Y_i is
binomially distributed (n_i, p_i). We assume that $Y_1, Y_2, \ldots, Y_k$ are
mutually independent random variables.

The magnitude of interest in this problem is easily evidenced
by the extensive literature devoted to it. In this regard, see
the survey paper by Harris (1977) and Section 10.4 of the book by
Mann, Schafer, and Singpurwalla (1974). In addition, the Defense
Advanced Research Projects Agency has recently issued a Handbook
for the Calculation of Lower Statistical Confidence Bounds on
System Reliability (1980).

Historically, the first significant work on this problem was
produced by Buehler (1957). However, Buehler's method as des-
cribed in that paper is difficult to implement computationally
when k>2.

We proceed by describing Buehler's method in Section 2. In Section 3 we specialize to series systems, that is, a system which fails whenever at least one subsystem fails. Sudakov's (1974) results are extended in Section 4 and employed to exhibit some optimality properties of the Lindstrom-Madden method (see Lloyd and Lipow (1962)) for constructing lower confidence bounds for the reliability of series systems of stochastically independent subsystems. Some numerical examples are given in Section 5 and the results needed for this generalization of Sudakov's Theorem are provided in the Appendix to this paper.

2. Buehler's Method for Lower Confidence Bounds

A system composed of k independent subsystems is said to be a coherent system (with respect to the specified decomposition into subsystems), if the system fails when all subsystems fail and the system functions when all subsystems function; and replacing a defective subsystem by a functioning subsystem can not cause a functioning system to fail. Coherent systems are described in Birnbaum, Esary and Saunders (1961) and Barlow and Proschan (1975).

To any system one can associate a function, $h(\tilde{p}) = h(p_1, p_2, \ldots, p_k)$, $0 \leq p_i \leq 1$, $i = 1, 2, \ldots, k$, where $h(\tilde{p})$ is the reliability of the system when p_i is the probability that the $i^{\underline{th}}$ subsystem functions. It is well-known that if the system is coherent,

$$0 \leq h(\tilde{p}) \leq 1 ,$$

$$h(0, \ldots, 0) = 0, \quad h(1, \ldots, 1) = 1 ,$$

and $h(p_1, \ldots, p_k)$ is non-decreasing in each variable.

For coherent systems, Buehler's method may be described as follows: The observed outcome $(y_1, \ldots, y_k)$ can assume any of $N = \prod_{i=1}^{k} (n_i+1)$ values, since $y_i = 0, 1, \ldots, n_i$. For convenience, we denote $n_i - y_i$ by x_i, $i = 1, 2, \ldots, k$.

A partition $(A_1, A_2, \ldots, A_s)$, $s > 1$, of the N possible outcomes is said to be a monotonic partition, that is, $A_1 < A_2 < \ldots < A_s$ if $(0, 0, \ldots, 0) \in A_1$, $(n_1, n_2, \ldots, n_k) \in A_s$ and if $\tilde{x}_1 = (x_{11}, \ldots, x_{1k})$, $\tilde{x}_2 = (x_{21}, \ldots, x_{2k})$ with $x_{1i} \leq x_{2i}$, $i = 1, 2, \ldots, k$, then $\tilde{x}_1 \in A_i$ implies $\tilde{x}_2 \in A_j$, $j \geq i$.

Let

$$f(\tilde{x}; \tilde{p}) = p_{\tilde{p}}(\tilde{X} = \tilde{x}) = \prod_{i=1}^{k} \binom{n_i}{x_i} p_i^{n_i - x_i} q_i^{x_i} = \prod_{i=1}^{k} \binom{n_i}{y_i} p_i^{y_i} q_i^{n_i - y_i} \qquad (2.1)$$

and for $1 \leq n \leq s-1$, let

$$a_n = \inf \left\{ h(p) \,\middle|\, \sum_{\tilde{x}_i \in A_i, i \leq n} f(\tilde{x}_i; \tilde{p}) = \alpha \right\} \qquad (2.2)$$

and $a_s = 0$.

Each such partition may be identified with a function defined on the set of sample outcomes by defining the ordering function $g(\tilde{x})$, where

$$g(\tilde{x}) = n \quad \text{if} \quad \tilde{x} \in A_n , \quad 1 \leq n \leq s ; \qquad (2.3)$$

obviously $g(\tilde{x})$ inherits the monotonicity properties of the partition.

Subsequently it will be convenient to use ordering functions $g(\tilde{x})$ such that the range of $g(\tilde{x})$ will be a finite set of real numbers, $r_1 < r_2 < \ldots < r_s$. With no loss of generality, we can identify the sets A_i by defining $A_i = \left\{ \tilde{x} \mid g(\tilde{x}) = r_i \right\}$, $i = 1, 2, \ldots, s$. We can now establish the following theorems.

Theorem 2.1. Let $\tilde{X}$ be distributed by (2.1). Then $a_{g(\tilde{X})}$ is a $(1-\alpha)$ lower confidence bound for $h(\tilde{p})$. If $b_{g(\tilde{X})}$ is also a $(1-\alpha)$ lower confidence bound for $h(\tilde{p})$, then $b_i \leq a_i$, $1 \leq i \leq s$.

Proof: Fix $\tilde{p}$ and let $n(\tilde{p})$ be the smallest integer such that

$$P_{\tilde{p}}\left\{ \tilde{X} \ \epsilon \ \bigcup_{i=1}^{n(\tilde{p})} A_i \right\} \geq \alpha \ , \tag{2.4}$$

and

$$P_{\tilde{p}}\left\{ \tilde{X} \ \epsilon \ \bigcup_{i=n(\tilde{p})}^{s} A_i \right\} \geq 1-\alpha \ . \tag{2.5}$$

Let

$$D_n = \left\{ \tilde{p} \, | \, P_{\tilde{p}}\left\{ \tilde{X} \ \epsilon \ \bigcup_{i=1}^{n} A_i \right\} \geq \alpha \right\} \ . \tag{2.6}$$

Then $D_{g(\tilde{X})}$ is a $1-\alpha$ confidence set for $\tilde{p}$, since

$$P_{\tilde{p}}\left\{ \tilde{p} \ \epsilon \ D_{g(\tilde{X})} \right\} = P_{\tilde{p}}\left\{ g(\tilde{X}) \geq n(\tilde{p}) \right\} \geq 1-\alpha \ . \tag{2.7}$$

This establishes the first part of the conclusion. Further, since $h(\tilde{p})$ is continuous and $0 \leq p_i \leq 1$, the infimum in (2.2) is attained.

Now assume that i_1 is the smallest index such that $b_{i_1} > a_{i_1}$, $1 \leq i_1 \leq s-1$. Then, for some $\tilde{p}_0$, $\tilde{p}_1$,

$$b_{i_1} > \inf\left\{ h(\tilde{p}) \, | \, \sum_{x_i \epsilon A_i, \, i \leq i_1} f(\tilde{x}; \tilde{p}) = \alpha \right\} = h(\tilde{p}_0) \ ,$$

and

$$\sum_{x_i \epsilon A_i, \, i \leq i_1} f(\tilde{x}; \tilde{p}_1) > \alpha, \ h(\tilde{p}_1) < b_{i_1} \ .$$

Therefore

$$P_{\tilde{p}_1}\left\{ h(\tilde{p}_1) < b_{g(\tilde{X})} \right\} \geq \sum_{\tilde{x}_i \epsilon A_i, \, i \leq i_1} f(\tilde{x}; \tilde{p}_1) > \alpha,$$

a contradiction.

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<u>Remark</u>. Let $d_n = \sup\left\{1-h(\tilde{p}) \Big|_{x_i \in A_i, i \le n} \sum f(\tilde{x}_i;\tilde{p}) = \alpha\right\}$. Then d_n is a

$(1-\alpha)$ upper confidence bound for $1-h(\tilde{p})$, the unreliability.

Let $A = \left\{\tilde{x} \in E_k, \ 0 \le x_i < a_i, \ i=1,2,\ldots,k\right\}$ and let $g(\tilde{x})$ be contin-

uous on $\overline{A}$ (the closure of A) and strictly increasing in each

variable for $\tilde{x} \in A$. $g(\tilde{x})$ is to be regarded as an ordering function

as described immediately preceding Theorem 2.1. We require the

following additional property of $g(\tilde{x})$.

Fix $\tilde{x}_o \in A$. Let $g(\tilde{x}_o) < g(a_1,0,\ldots,0) = g_1$. Then

$g(y_1,0,\ldots,0) = g(\tilde{x}_o)$ has a unique solution in y_1. Proceeding

recursively, let $i_1 \le y_1$ and define $y_2 = y_2(i_1)$ as the solution

of $g(\tilde{x}_o) = g(i_1,y_2,0,\ldots,0)$. For each $1 \le j \le k$ and $i_{j-1} \le y_{j-1}$,

$i_{j-2} \le y_{j-2},\ldots, \ i_1 \le y_1$, let $y_j = y_j(i_1,i_2,\ldots,i_{j-1})$ be the

solution of

$$g(\tilde{x}_o) = g(i_1,i_2,\ldots,i_{j-1},y_j,0,\ldots,0) \ . \tag{2.8}$$

We require that the equations indicated in (2.8) have

unique solutions for each y_j.

Then define

$$F(\tilde{x}_o;\tilde{p}) = \sum_{i_1=0}^{[y_1]} \sum_{i_2=0}^{[y_2]} \cdots \sum_{i_k=0}^{[y_k]} f(\tilde{i};\tilde{p}) \ , \tag{2.9}$$

where, for $j>1$, $y_j = y_j(i_1,i_2,\ldots,i_{j-1})$. Let

$$f^*(\tilde{x}_o;a) = \sup_{h(\tilde{p})=a} F(\tilde{x}_o;\tilde{p}) \ , \quad 0<a<1 \ . \tag{2.10}$$

Then we have

<u>Theorem 2.2</u>. If $\tilde{x}_o$ satisfies $\inf_{0<a<1} f^*(x_o;a) = 0$, $\sup_{0<a<1} f^*(x_o;a) = 1$

and $f^*(x_o;a)$ is a strictly increasing function of a, and if

$\tilde{x}_o \in A_n$ where $g(\tilde{x})$ determines $(A_1,A_2,\ldots,A_s)$, and if

$$b = \inf\left\{ h(\tilde{p}) \,\Big|\, \sum_{x \in A_i, 1 \le n} f(\tilde{x}_i, \tilde{p}) = \alpha \right\} , \qquad (2.11)$$

then we have

$$f^*(\tilde{x}_o; b) = \alpha .$$

Proof: Since the infimum in (2.11) is attained, there is a $\tilde{p}_o$ such that $b = h(\tilde{p}_o)$ and $F(\tilde{x}_o; \tilde{p}_o) = \alpha$. Then $f^*(\tilde{x}_o, b) \ge \alpha$. If $f^*(\tilde{x}_o, b) > \alpha$, there exists $\tilde{p}_a$, with $a = h(\tilde{p}_a)$, $a < b$ and $f^*(\tilde{x}_o; a) = \alpha$ contradicting (2.11).

Obviously, the above discussion can easily be modified to obtain upper confidence bounds on the unreliability $1 - h(\tilde{p})$ by replacing inf by sup in (2.11) and requiring that $f^*(\tilde{x}_o; a)$ be a strictly decreasing function of a, $0 < a < 1$.

3. Applications to Series Systems

For a series system $h(\tilde{p}) = \prod_{i=1}^{k} p_i$. Further, throughout this section we assume that $g(\tilde{x})$ satisfies the conditions necessary to insure that the solutions for $y_1, \ldots, y_k$ indicated in (2.8) are unique. Then we have the following theorem.

Theorem 3.1. If $h(\tilde{p}) = \prod_{i=1}^{k} p_i$, then $\inf_{0 < a < 1} f^*(\tilde{x}_o; a) = 0$, $\sup_{0 < a < 1} f^*(\tilde{x}_o, a) = 1$ and $f^*(\tilde{x}_o; a)$ is strictly increasing in a, whenever $\tilde{x}_o = (x_{o1}, \ldots, x_{ok})$ satisfies $x_{oj} < n_j$, $j = 1, 2, \ldots, k$.

Proof. Since $h(\tilde{p}) = 1$ if and only if $p_i = 1$, $i = 1, 2, \ldots, k$, it follows from (2.1) that

$$\lim_{\substack{a \to 1 \\ h(\tilde{p}) = a}} \sup F(\tilde{x}_o; \tilde{p}) = 1 .$$

Similarly, $h(\tilde{p}) = 0$ if and only if at least one $p_i = 0$, $i=1,2,\ldots,k$. Since $F(\tilde{x}_0;\tilde{p}) \leq P_{\tilde{p}}\{X_i < n_i\} = 1 - P_{\tilde{p}}\{X_i = n_i\} = 1 - q_i^{n_i}$, we have

$$\lim_{\substack{a \to 0 \\ h(\tilde{p}) = a}} \sup F(\tilde{x}_0;\tilde{p}) = 0 .$$

To show that $f^*(\tilde{x}_0;a)$ is strictly increasing in a, consider $0 < a < b < 1$ and let $\tilde{p}_a = (p_{a1},\ldots,p_{ak})$ satisfy $f^*(\tilde{x}_0;a) = F(\tilde{x}_0;\tilde{p}_a)$. Similarly, let $\tilde{p}_b$ satisfy $f^*(\tilde{x}_0;b) = F(\tilde{x}_0;\tilde{p}_b)$. Let $I = \{i_1,i_2,\ldots,i_r\}$ be any non-empty set of indices such that $p_{ai_j}(\frac{b}{a})^{1/r} < 1$ and let I^c be the remaining indices. Then

$$\left(\prod_{j \in I} p_{ai_j}(\frac{b}{a})^{1/r} \right) \prod_{j \in I^c} p_{ai_j} = b . \tag{3.1}$$

From the monotone likelihood ratio property of the binomial distribution,

$$F(\tilde{x}_0;\tilde{p}_a) < F(\tilde{x}_0;\tilde{p}^*) ,$$

where the components of p^* are given by (3.1). Then

$$F(\tilde{x}_0;\tilde{p}^*) \leq \sup_{h(\tilde{p}) = b} F(\tilde{x}_0;\tilde{p}) = F(\tilde{x}_0;\tilde{p}_b) = f^*(x_0;b) .$$

4. Sudakov's Method

Let

$$I_p(r,s) = \frac{1}{B(r,s)} \int_0^p t^{r-1}(1-t)^{s-1}dt .$$

Then if y is an integer, $y < n$, we have

$$\sum_{i=0}^{y} \binom{n}{i} p^{n-i}q^i = I_p(n-y,y+1) .$$

For $0 \leq y < n$, real, define $u(n,y,\alpha)$ by $\alpha = I_{u(n,y,\alpha)}(n-y,y+1)$.

Thus, for integer values of y, $u(n,y,\alpha)$ is a $100(1-\alpha)$ percent

lower confidence limit for p. Sudakov (1973) showed that for

$n_1 \leq n_2 \leq \ldots \leq n_k$ and $g(\tilde{x}) = \prod_{i=1}^{k} (n_i - x_i)$,

$$u(n_1, y_1, \alpha) \leq b \leq u(n_1, [y_1], \alpha) ,$$

where $\qquad y_1 = n_1 q_0, \quad q_0 = 1 - \prod_{i=1}^{k} ((n_i - x_{oi})/n_i) .$

$u(n_1, y_1, \alpha)$ is called the Lindstrom-Madden method for determining

lower confidence limits for the reliability of series systems

(see Lloyd and Lipow (1962)).

Lipow and Riley (1959) used a different ordering function;

nevertheless they noted that for "small" n_i, their tabulated

values provided good agreement with the results   using the

Lindstrom-Madden method. For large values of n_i, the tabulated

values that they provided are based on the Lindstrom-Madden

method. Here we provide a further justification for the

Lindstrom-Madden method by establishing that it provides conserva-

tive lower confidence limits (i.e. is a lower bound to b defined

in (2.9)) using the ordering function $g(\tilde{x})$ employed by Sudakov

and we also obtain an upper bound for b, thus determining the

possible error of the Lindstrom-Madden method.

Sudakov's proof is unnecessarily complicated and contains

some incorrect assertions, which nevertheless do not affect the

validity of the conclusion. In the Appendix we provide a simpler

proof of some auxiliary results needed for the generalization of

Sudakov's theorem given below.

<u>Theorem 4.1.</u> Let $g(\tilde{x})$ satisfy the hypothesis of Theorem 3.1.
Then,

$$b \leq \min_{1 \leq i \leq k} u(n_1, [y_i^*], \alpha) , \tag{4.1}$$

where b is given by (2.11) and $y_i^* = y_i(j_1, j_2, \ldots, j_{i-1})$ is evaluated
at $j_\ell = 0$, $\ell = 1, 2, \ldots, i-1$.   Note that $y_1 = y_1^*$. If we also have

$$\frac{y_j - i_j}{n_j - i_j} \geq \frac{y_{j+1}}{n_{j+1}} , \quad j = 1, 2, \ldots, k-1 , \tag{4.2}$$

then

$$u(n_1, y_1, \alpha) \leq b . \tag{4.3}$$

<u>Proof:</u> (4.1) is immediate from (2.11) upon setting $p_j = 1$, $j \neq i$ and
solving $F(\tilde{x}_o; 1, \ldots, 1, p_i, 1, \ldots, 1) = \alpha$. Recall that $n_1 \leq n_2 \leq \cdots \leq n_k$
and

$$F(\tilde{x}_o; \tilde{p}) = \sum_{i_1=0}^{[y_1]} b(n_1 - i_1; p_1, n_1)$$

$$\cdots \sum_{i_{k-1}=0}^{[y_{k-1}]} b(n_{k-1} - i_{k-1}; p_{k-1}, n_{k-1}) I_{p_k}(n_k - [y_k], [y_k]+1) . \tag{4.4}$$

Now, apply Lemmas A1, A2, and A3 to the innermost sum in (4.4), to get

$$\sum_{i_{k-1}=0}^{[y_{k-1}]} b(n_{k-1} - i_{k-1}; p_{k-1}, n_{k-1}) I_{p_k}(n_k - [y_k], [y_k]+1) \leq$$

$$\sum_{i_{k-1}=0}^{[y_{k-1}]} b(n_{k-1} - i_{k-1}; p_{k-1}, n_{k-1}) I_{p_k}(n_k - y_k, y_k+1) \leq$$

$$\sum_{i_{k-1}=0}^{[y_{k-1}]} b(n_{k-1} - i_{k-1}; p_{k-1}, n_{k-1}) I_{p_k}(n_{k-1} - y_{k-1}, y_{k-1} - i_{k-1}+1) \leq$$

$$I_{p_{k-1} p_k}(n_{k-1} - y_{k-1}, y_{k-1}+1) .$$

Repeated applications of the above establish that

$$F(\tilde{x}_o; \tilde{p}) \leq I_{\underset{i=1}{\overset{k}{\Pi} p_i}} (n_1 - y_1, y_1 + 1) . \qquad (4.5)$$

(4.3) follows immediately from (4.5), completing the proof.

Remarks. If (4.3) holds and y_1 is an integer, then $b = f(n_1, y_1, \alpha)$.

It has often been suggested (Lloyd and Lipow (1962), Winterbottom (1974), Bolshev and Loginov (1966), Mirniy and Solov'yev (1964)) that the confidence level should depend only on n_1, the smallest sample size. We now provide a numerical illustration to show that the bound in (4.1) may be improved by taking all the n_i's into consideration.

Let $k=3$, $\alpha = .1$, $\tilde{n} = (10,12,30)$, $\tilde{x}_o = (0,3,0)$. Then for $g(\tilde{x}) = \overset{3}{\underset{i=1}{\Pi}} (n_i - x_i)$, $f(n_1, [y_1], \alpha) = .541$, $f(n_2, [y_2], \alpha) = .525$, $f(n_3, [y_3], \alpha) = .639$. The use of (4.3) establishes $.500 \leq b \leq .525$.

Note that if $x_{oi} = n_i$, for some i, $1 \leq i \leq k$, then $g(\tilde{x}) = 0$ and $b=0$. It seems reasonable to use $b=0$ as the lower confidence limit whenever $x_{oi} = n_i$ for any monotone ordering function satisfying the conditions of Section 2.

We now show that if $g(\tilde{x}) = \overset{k}{\underset{i=1}{\Pi}} (n_i - x_i)$, then (4.2) is satisfied and Theorem 4.1 applies. This result will extend a result due to Winterbottom (1974), who established this fact for particular special cases. In addition, we will also show that (4.2) holds for a number of other ordering functions used in the literature.

Theorem 4.2. Let $g(\tilde{x}) = \prod_{i=1}^{k} (n_i - x_i + \alpha_i)$, where $\alpha_i \geq 0$ and $n_{i+1}\alpha_i \geq \alpha_{i+1}n_i$, $i=1,2,\ldots,k-1$. Then (4.2) is satisfied.

Proof. If

$$(n_i - y_i + \alpha_i) \prod_{j=i+1}^{k} (n_j + \alpha_j) = c$$

and

$$(n_i - k_i + \alpha_i)(n_{i+1} - y_{i+1} + \alpha_{i+1}) \prod_{j=i+2}^{k} (n_j + \alpha_j) = c \;,$$

then we have

$$(n_i - y_i + \alpha_i)(n_{i+1} + \alpha_{i+1}) = (n_i - k_i + \alpha_i)(n_{i+1} - y_{i+1} + \alpha_{i+1}) \;,$$

establishing

$$\frac{y_i - k_i}{n_i - k_i} = \frac{y_{i+1}}{n_{i+1}} \frac{n_{i+1}(n_i + \alpha_i - k_i)}{(n_{i+1} + \alpha_{i+1})(n_i - k_i)} \;.$$

Thus (4.2) holds if

$$\frac{n_{i+1}(n_i + \alpha_i - k_i)}{(n_{i+1} + \alpha_{i+1})(n_i - k_i)} \geq 1 \;;$$

this last inequality will be true whenever $n_{i+1}\alpha_i \geq \alpha_{i+1}n_i$. In particular, this is valid when $\alpha_i = 0$, $i = 1, \ldots, k$ which is Sudakov's ordering function.

Theorem 4.3. If $g(\tilde{x}) = 1 - \sum_{i=1}^{k} x_i / n_i$, then (4.2) is satisfied.

Proof. If $1 - y_i/n_i = c = 1 - \dfrac{k_i}{n_i} - \dfrac{y_{i+1}}{n_{i+1}}$, then

$$\frac{y_i - k_i}{n_i} = \frac{y_{i+1}}{n_{i+1}}$$

or

$$\frac{y_i - k_i}{n_i - k_i} > \frac{y_{i+1}}{n_{i+1}} \;.$$

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This type of ordering function has been employed by Pavlov (1973), for example.

__Theorem 4.4.__ Let $g(\tilde{x}) = \sum_{i=1}^{k} a_i x_i + z_\alpha (a_i^2 x_i)^{\frac{1}{2}}$, where z_α satisfies $1 - \phi(z_\alpha) = \alpha$ and $\phi(x)$ is the standard normal distribution function, $a_1 \geq a_2 \geq \cdots \geq a_k$, and $a_i = (n_i \sum_{i=1}^{k} 1/n_i)^{-1}$. Then $g(\tilde{x})$ satisfies (4.2) if and only if

$$(a_j - a_{j+1}) y_j \geq (a_j - a_{j+1}) z_\alpha^2 + a_j k_j - \alpha_j k_j (z_\alpha^2 a_j + 2c - a_j (y_j + k_j)) . \qquad (4.6)$$

__Proof:__ If $g(\tilde{x}_0) = c + \sum_{i=1}^{j-1} a_i k_i$, then defining $\sum_{i=1}^{j-1} a_i^2 k_i = c_1$,

$$a_j y_j + z_\alpha (c_1 + a_j^2 y_j)^{\frac{1}{2}} = c \qquad (4.7)$$

and

$$a_j k_j + a_{j+1} y_{j+1} + z_\alpha (c_1 + a_j^2 k_j + a_{j+1}^2 y_{j+1})^{\frac{1}{2}} = c . \qquad (4.8)$$

Equating the left hand sides of (4.7) and (4.8), we obtain (4.6). If $k=2$, (4.6) holds for all cases of interest.

If (4.6) holds, then setting

$$1 - \alpha = (\Gamma(x))^{-1} \int_0^{f(x, 1-\alpha)} t^{x-1} e^{-t} dt ,$$

a straightforward limiting argument shows that

$$\max_i a_i f([y_i] + 1, 1 - \alpha) \leq b \leq a_1 f(y_1 + 1, 1 - \alpha) . \qquad (4.9)$$

This ordering function has been used by Johns (1976) and b in (4.7) is the value tabulated by Johns for $k=2$. The validity of the lower bound does not depend on (4.6). In Table 1 below, the lower and upper bounds given in (4.9) are tabulated along with the values given by Johns for $\alpha = .1$. These refer to upper confidence limits for the Poisson parameter combinations $a_1 \lambda_1 + a_2 \lambda_2$.

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Note in particular that three of the values tabulated by
Johns (indicated by asterisks) violate (4.9). Specifically consider
5.24, in which case $[y_1] = 5$, since $g^*(2,5) = 4.78$, $g^*(5,0) = 4.72$
and $g^*(6,0) = 5.48$. Using the Poisson approximation we obtain the
value 9.275 for the upper confidence limit to λ for $\alpha = .1$ and thus
$a_1\lambda_1 + a_2\lambda_2 = 5.56$. Consequently the sup must exceed 5.56. An
alternative approach to the one suggested by Johns for $k \geq 3$ is to
simply use $a_1 f(y_1+1, 1-\alpha)$ for b.

<u>Table 1</u>

Comparison of Upper and Lower Bounds
With Values Tabulated by Johns for $\alpha = .1$

a_1	x_1	x_2	Lower Bound	Upper Bound	Johns' Tabled Value
.9	7	2	4.79	5.50	5.17
.9	3	0	2.07	2.27	2.16
.75	6	3	6.00	6.65	6.23
.75	12	3	7.90	8.29	7.91
.67	3	3	5.36	5.61	5.33*
.67	15	2	8.71	9.24	8.81
.60	5	2	5.56	5.62	5.24*
.60	7	6	9.24	9.53	9.18*

5. Numerical Examples and Concluding Remarks

Examples 1 and 2 illustrate the method we have described in
this paper.

<u>Example 1</u>: Let $H(\tilde{x}) = \prod_{i=1}^{k} (n_i - x_i)$, $\alpha = .05$, $k = 5$, $\tilde{n} =$
$(20,30,40,25,60)$, $\tilde{x} = (2,6,10,8,15)$. Then the 95% upper confi-
dence limit for the failure probability is contained in $(.86, .88)$.

<u>Example 2</u>: Let $H(\tilde{x}) = \Pi(n_i - x_i)$, $\alpha = .05$, $k = 2$, $\tilde{n} = (10,10)$,
$\tilde{x} = (3,2)$. Then the 95% upper confidence limit for the failure
probability is contained in (.70, .73). The value given in Lipow
and Riley (1959) is .70.

<u>Remarks</u>. In this paper we have showed that the Lindstrom-Madden
technique is conservative for ordering functions satisfying (4.2).
Further, if y_1 is an integer, then the Lindstrom-Madden method is
exact. We have also relaxed the conditions needed in Winterbottom
(1974) and provided an alternative to the method of Johns (1976).

Appendix

The auxiliary results employed in the proof of Theorem 4.1 are provided here.

Lemma A1: $I_y(n-x, x+1)$, $0 \leq y \leq 1$, is a decreasing function of n and an increasing function of x. $I_y(np, nq+1)$, $p+q = 1$, $0 < p < 1$, is an increasing function of q.

Proof: The proof is immediate from the observation that the beta distribution with parameters α and β has monotone likelihood ratio in α and $-\beta$ and that if a probability distribution has monotone likelihood ratio in θ, $F_\theta(x)$ is a decreasing function of θ (Lehmann (1959), p. 68 and p. 74).

Lemma A2: If $\dfrac{y_i - k_i}{n_i - k_i} \geq \dfrac{y_{i+1}}{n_{i+1}}$ and $n_i \leq n_{i+1}$, then

$$I_y(n_i - y_i, y_i - k_i + 1) \geq I_y(n_{i+1} - y_{i+1}, y_{i+1} + 1) . \qquad (A.1)$$

Proof: Rewriting the left and right hand sides of (A.1) as

$$I_y\left[(n_i - k_i)(1 - \frac{y_i - k_i}{n_i - k_i}), \ (n_i - k_i)(\frac{y_i - k_i}{n_i - k_i}) + 1\right] \geq$$

$$I_y\left[n_{i+1}(1 - \frac{y_{i+1}}{n_{i+1}}), \ n_{i+1}(\frac{y_{i+1}}{n_{i+1}}) + 1\right] , \qquad (A.2)$$

Lemma A1 applies and the conclusion follows.

Lemma A3: Let $y_1 y_2 = y$, $0 \leq y_i \leq 1$, $i = 1, 2$. Then

$$I_{y_1 y_2}(n-x, x+1) \geq \sum_{k=0}^{[x]} b(n-k; y_1, n_1) I_{y_2}(n-x, x-k+1) . \qquad (A.3)$$

Proof:

$$\sum_{k=0}^{[x]} \binom{n}{k} y_1^{n-k}(1-y_1)^k \frac{\Gamma(n-k+1)}{\Gamma(n-x)\Gamma(x-k+1)} \int_0^{y_2} t^{n-x-1}(1-t)^{x-k}dt$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-x)} \sum_{k=0}^{[x]} \frac{(1-y_1)^k y_1^{n-k}}{k!\Gamma(x-k+1)} \int_0^{y_1 y_2} \left(\frac{t}{y_1}\right)^{n-x-1} \left(\frac{y_1-t}{y_1}\right)^{x-k} \frac{dt}{y_1}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-x)} \int_0^{y_1 y_2} \sum_{k=0}^{[x]} \frac{(1-y_1)^k t^{n-x-1}(y_1-t)^{x-k}}{k!\Gamma(x-k+1)} dt \; .$$

Thus (A.3) will hold whenever

$$\frac{\Gamma(n+1)}{\Gamma(n-x)\Gamma(x+1)} \int_0^{y_1 y_2} t^{n-x-1}(1-t)^x dt \geq$$

$$\frac{\Gamma(n+1)}{\Gamma(n-x)} \int_0^{y_1 y_2} \sum_{k=0}^{[x]} \frac{(1-y_1)^k t^{n-x-1}(y_1-t)^{x-k}}{k!\;\Gamma(x-k+1)} dt$$

or

$$\frac{\Gamma(n+1)}{\Gamma(r-x)\Gamma(x+1)} \int_0^{y_1 y_2} t^{n-x-1}(1-t)^x \cdot$$

$$\left(1 - \sum_{k=0}^{[x]} \frac{\Gamma(x+1)}{k!\Gamma(x-k+1)} \left(\frac{1-y_1}{1-t}\right)^k \left(\frac{y_1-t}{1-t}\right)^{x-k} \right) dt \geq 0 \; . \tag{A.4}$$

Writing $\frac{y_1-t}{1-t} = \left(1 - \frac{1-y_1}{1-t}\right)$ and noting that $0 \leq t \leq y_1 y_2 \leq 1$, $t < y_1$ and $(1-t) > 1-y_1$, we observe that (A.4) holds and the lemma is proved.

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REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER #2093	2. GOVT ACCESSION NO. AD-A089 666	3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle) BOUNDS FOR OPTIMAL CONFIDENCE LIMITS FOR SERIES SYSTEMS	5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s) Bernard Harris and Andrew P. Soms	8. CONTRACT OR GRANT NUMBER(s) N00014-79-C-0321 DAAG29-75-C-0024 DAAG29-80-C-0041

9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 4 - Statistics & Probability

11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below	12. REPORT DATE June 1980
	13. NUMBER OF PAGES 18

14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Lindstrom-Madden approximation; Optimal confidence bounds;
Reliability; Series system.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
    Lindstrom-Madden type approximations to the lower confidence
limit on the reliability of a series system are theoretically
justified by extending and simplifying the results of Sudakov (1973).
Applications are made to Johns (1976) and Winterbottom (1974).
Numerical examples are presented.